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Conte Truncated Expansion and Applications

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In the special Conte truncated expansion approach one obtains different solutions of the Prigogine–Lefever equation by use of various solutions of a type of Riccati equation, including the periodic soliton solutions and singular soliton solutions. In order to acquire conveniently the soliton solutions of the Boussinesq equation, a proper transformation is applied. Using the special Conte truncated expansion approach yields the known bell-shape solutions and some new soliton solutions like $\cot^2 \times \sec^2$, $\tan^2 \times c\sec^2$, $\tanh^2 \times \mathrm{sech}^2$, etc. We also study the soliton solutions of the modified Burgers equation (MBE). Using leading term analysis, we find the exponent is a fraction, i.e., $-\frac{1}{2}$. Therefore, the special Conte truncated expansion approach cannot be used directly. A transformation is first made to them another form of the MBE. Various soliton solutions of MBE are then presented, including the periodic solutions and singular soliton solutions.

KEY WORDS: truncate expansion; exact solution; Riccati equation.

1. INTRODUCTION

As we know, the Painleve analysis developed by Weiss–Tabor–Carnevale (WTC) is a powerful tool to find soliton solutions of the nonlinear evolution equations. Of course, it is also an active method to prove the integrability of some hierarchies of evolution equations. In 1989, Conte proposed an invariant version of the WTC approach (Weiss *et al.*, 1983). Later, Pickering proposed a nonstandard truncation approach basing on the Conte invariant Painleve analysis (Conte, 1989). Similar to Conte's consideration, Lou acquired some types of expansions to study

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the Painleve properties, which can be used to obtain some new exact solutions of nonlinear evolution equations (Chen and Lou, 2003; Lou *et al.*, 1991; Lou and Ni, 1989). In this paper, we use a special Conte expansion to get some soliton solutions of three kinds of nonlinear evolution equations. First we look back on Pickering's general expansion.

Given an evolution equation, say, in two independent variables

$$U(\Psi, \Psi_x, \Psi_t, \Psi_{xx}, \Psi_{tt}, \ldots) = 0, \tag{1}$$

let

$$\Psi = \sum_{j=0}^{\infty} \Psi_j \xi^{j+\alpha}, \tag{2}$$

with arbitrary Ψ_i, ξ being determined by

$$\xi_x = \sum_{j=0}^{N} S_j \xi^j, \quad \xi_t = \sum_{j=0}^{N} Y_j \xi^j. \tag{3}$$

In (3), $2N + 2$ functions S_j, Y_j satisfy $2N - 1$ consistent conditions as follows

$$\begin{cases} S_{jt} - Y_{jx} + \sum\limits_{n=1}^{j+1} n(S_n Y_{j+1-n} - Y_n S_{j+1-n}) = 0, \; j = 0, 1, 2, \ldots, N, \\ \sum\limits_{n=j+1-N}^{N} (S_n Y_{j+1-n} - Y_n S_{j+1-n}) = 0, \; j = N+1, \ldots, 2N-2. \end{cases} \tag{4}$$

When we take $N = 2$ in (3), the general expansion (2) with (3) is just Pickering's expansion (Chen and Lou, 2003). Taking $N = 2$ in (3), we fix the expansion function as

$$\xi \equiv g = \lambda - \mu T, \quad T = \left(\frac{\Psi_x}{\Psi} - \frac{\Psi_{xx}}{2\Psi_x} \right)^{-1}, \tag{5}$$

with λ, μ being arbitrary constants.

When we take $\lambda = 0$, $\mu \neq 0$, (2) with (5) is reduced back the usual Conte expansion. In terms of the special selection (5), (3) becomes

$$g_x = \mu + \frac{1}{2\mu} s g^2, \quad g_t = -c + c_x g - \frac{1}{2\mu}(c_{xx} + cs)g^2, \tag{6}$$

where $s = \frac{3}{2}(\frac{\Psi_{xx}}{\Psi_x})^2 - \frac{\Psi_{xxx}}{\Psi_x}$, $c \equiv -\frac{\Psi_t}{\Psi_x}$. It is easy to verify that (6) is the Mobius transformation invariants (Chen and Lou, 2003). The corresponding compatibility condition (4) reduces to a singe one

$$s_t = -c_{xxx} - 2sc_x - s_x c.$$

For the convenience of calculations, taking $u = 1$ in (6), we have

$$g_x = 1 + \frac{1}{2}sg^2, \quad g_t = -c + c_x g - \frac{1}{2}(c_{xx} + cs)g^2, \tag{7}$$

which is a type of Riccati equation.

The expasion form (2) with (7) is called the special Conte truncated expansion. In particular, taking c and s are constants, (7) reduces to

$$g_x = 1 + \frac{1}{2}sg^2, \quad g_t = -c - \frac{1}{2}csg^2, \tag{8}$$

which has the following solutions.

$$\begin{cases} g = \sqrt{\frac{2}{s}}\tan\left(\sqrt{\frac{s}{2}}(x-ct)\right), s > 0, \\ g = \sqrt{\frac{2}{-s}}\tanh\left(\sqrt{\frac{-s}{2}}(x-ct)\right), s < 0, \end{cases} \begin{cases} g = \sqrt{\frac{2}{s}}\cot\left(\sqrt{\frac{s}{2}}(-x+ct)\right), s > 0, \\ g = \sqrt{\frac{2}{-s}}\coth\left(\sqrt{\frac{-s}{2}}(x-ct)\right), s < 0, \end{cases} \tag{9}$$

2. SOME APPLICATION

In what follows, we apply the expansion (2) with (8), (9) to three kinds of nonlinear evolution equations, which are the Prigogine–Lefever (PL) equation, the Boussinesq equation and the modified Burgers equation (MBE), to obtain soliton solutions. Using directly the Conte truncated expansion for the PL equation produces some new periodic solitons and singular soliton solutions. In order that soliton solutions of the Boussinesq equation can be obtained conveniently, a proper transformation is first applied. It is remarkable that solution forms like $\cot^2 \times \sec^2$, $\tan^2 \times c\sec^2$, $\tanh^2 \times \mathrm{sech}^2$, etc. are found for this equation. We find that the special Conte truncated expansion can not be used for the MBE. Therefore we make an appropriate transformtion. Then we have various soliton solutions to the MBE. The approach presented in this paper can be used generally.

Example 1. Prigogine and Lefever proposed a mathematical model in 1968 (Pickering, 1993), briefly called the PL equation

$$\begin{cases} u_t = Ku_{xx} + u^2 v - Bu, \\ v_t = Kv_{xx} - u^2 v + Bu, \end{cases} \tag{10}$$

where B is a constant, K denotes a dispersive coefficient. The system (10) systematically describes a biochemistry model (Weiss *et al.*, 1983). Let $u = \sum_{j=0}^{\infty} u_j q^{j+\alpha}$, $v = \sum_{j=0}^{\infty} v_j q^{j+\alpha}$ be a solution of (10), then the leading terms are $u_0 q^{\alpha}$, $v_0 q^{\beta}$. Balancing the linear terms of the highest order u_{xx}, v_{xx} with the nonlinear terms in Eq. (10) yields that $\alpha - 2 = 2\alpha + \beta$, $\beta - 2 = 2\alpha + \beta$, i.e., $\alpha = -1$, $\beta = -1$.

Therefore, let

$$u = \frac{u_0}{q} + u_1 + u_2 q,\ v = \frac{v_0}{q} + v_1 + v_2 q,\ q_x = 1 + \frac{1}{2}sq^2,\ q_t = -c - \frac{1}{2}csq^2, \tag{11}$$

where $u_0, u_1, u_2, v_0, v_1, v_2$ are to be determined. It is easy to have that

$$\begin{cases} u_t = \frac{u_0 c}{q^2} + \frac{1}{2}u_0 cs - u_2 c - \frac{1}{2}csu_2 q^2, \\ u_{xx} = \frac{2u_0}{q^3} + \frac{su_0}{q} + u_2 sq + \frac{1}{2}u_2 s^2 q^3, \\ v_t = \frac{v_0 c}{q^3} + \frac{1}{2}v_0 cs - v_2 c - \frac{1}{2}csv_2 q^2, \\ v_{xx} = \frac{2v_0}{q^3} + \frac{sv_0}{q} + v_2 sq + \frac{1}{2}v_2 s^2 q^3. \end{cases} \tag{12}$$

Substituting (12) and (11) into (10), and setting all the coefficients of different powers of q to zero give rise to a set of overdetermined equations

$$\begin{cases} 2u_0 K + u_0^2 v_0 = 0, u_0 c = 2u_0 u_1 v_0 + u_0^2 v_1, \\ su_0 K + u_1^2 v_0 + 2u_0 v_0 u_2 + 2u_0 u_1 v_1 + u_0^2 v_2 - Bu_0 = 0, \\ \frac{1}{2}u_0 cs - u_2 c = u_1^2 v_1 + 2u_0 u_2 v_1 + 2u_0 u_1 v_2 - Bu_1 + 2u_1 u_2 v_0 = 0, \\ Ku_2 s + u_2^2 v_0 + 2u_1 u_2 v_1 + u_1^2 v_2 + 2u_0 u_2 v_2 - Bu_2 = 0, \\ -\frac{1}{2}csu_2 = u_2^2 v_1 + 2u_1 u_2 v_2, \frac{1}{2}Ku_2 s^2 + u_2^2 v_2 = 0, \\ 2v_0 K - u_0^2 v_0 = 0, v_0 c = -2u_0 u_1 v_0 - u_0^2 v_1, \\ sv_0 K - u_1^2 v_0 - 2u_0 v_0 u_2 - 2u_0 u_1 v_1 - u_0^2 v_2 + Bu_0 = 0, \\ \frac{1}{2}v_0 cs - v_2 c = -u_1^2 v_1 - 2u_1 u_2 v_0 - 2u_0 v_1 u_2 + Bu_1 - 2u_1 v_2 u_0 = 0, \\ Kv_2 s - u_2^2 v_0 - 2u_1 u_2 v_1 - u_1^2 v_2 - 2u_0 u_2 v_2 + Bu_2 = 0, \\ -\frac{1}{2}scv_2 = -u_2^2 v_1 - 2u_1 u_2 v_2, \frac{1}{2}Kv_2 s^2 - u_2^2 v_2 = 0. \end{cases} \tag{13}$$

In the case of $u_0 \neq 0$, $v_0 \neq 0$, solving Eqs. (13) gives

$$u_0 = \sqrt{2K}\epsilon,\ v_0 = -\sqrt{2K}\epsilon,\ u_2 = \begin{cases} \sqrt{\frac{K}{2}}s, & s > 0, \\ -\sqrt{\frac{K}{2}}s, & s < 0, \end{cases} \quad v_2 = \begin{cases} -\sqrt{\frac{K}{2}}s, & s > 0, \\ \sqrt{\frac{K}{2}}s, & s < 0, \end{cases}$$

where $\epsilon = \pm 1$.

When $s > 0$, we have $\epsilon = 1$, $v_1 = 0$, $u_1 = -\frac{\sqrt{2K}c}{4K}$; $\epsilon = -1$, $v_1 = -\frac{\sqrt{2K}c}{4K}$; $u_1 = 0$. When $s < 0$, we have $u_1 = -\frac{c}{4\sqrt{2K}\epsilon} + \frac{c}{4\sqrt{2K}}$, $v_1 = \frac{c}{2\sqrt{2K}\epsilon} + \frac{c}{2\sqrt{2K}}$.

Case 1. When $s > 0$, we have the following periodic soliton solutions

(i) $$\begin{cases} u = \sqrt{Ks}\cot\left(\sqrt{\frac{s}{2}}(x - ct)\right) + \sqrt{Ks}\tan\left(\sqrt{\frac{s}{2}}(x - ct)\right) - \frac{\sqrt{2K}c}{2K}, \\ v = -\sqrt{Ks}\cot\left(\sqrt{\frac{s}{2}}(x - ct)\right) - \sqrt{Ks}\tan\left(\sqrt{\frac{s}{2}}(x - ct)\right), \end{cases}$$

(ii) $$\begin{cases} u = -\sqrt{Ks}\cot\left(\sqrt{\frac{s}{2}}(x-ct)\right) + \sqrt{Ks}\tan\left(\sqrt{\frac{s}{2}}(x-ct)\right), \\ v = \sqrt{Ks}\cot\left(\sqrt{\frac{s}{2}}(x-ct)\right) - \sqrt{Ks}\tan\left(\sqrt{\frac{s}{2}}(x-ct)\right) - \frac{\sqrt{2K}c}{2K}, \end{cases}$$

(iii) $$\begin{cases} u = \sqrt{Ks}\tan\left(\sqrt{\frac{s}{2}}(x-ct)\right) + \sqrt{Ks}\cot\left(\sqrt{\frac{s}{2}}(x-ct)\right) - \frac{\sqrt{2K}c}{4K}, \\ v = -\sqrt{Ks}\tan\left(\sqrt{\frac{s}{2}}(x-ct)\right) - \sqrt{Ks}\tan\left(\sqrt{\frac{s}{2}}(x-ct)\right), \end{cases}$$

(iv) $$\begin{cases} u = -\sqrt{Ks}\tan\left(\sqrt{\frac{s}{2}}(-x+ct)\right) + \sqrt{Ks}\cot\left(\sqrt{\frac{s}{2}}(-x+ct)\right), \\ v = \sqrt{Ks}\tan\left(\sqrt{\frac{s}{2}}(-x+ct)\right) - \sqrt{Ks}\tan\left(\sqrt{\frac{s}{2}}(x-ct)\right) - \frac{\sqrt{2K}c}{2K}. \end{cases}$$

Case 2. When $s < 0$, we obtain the singular soliton solutions as follows

(v) $$\begin{cases} u = \sqrt{-Ks}\epsilon\coth\left(\sqrt{\frac{-s}{2}}(x-ct)\right) + \sqrt{-Ks}\tanh\left(\sqrt{\frac{-s}{2}}(x-ct)\right) - \frac{c}{4\sqrt{2K}\epsilon}, \\ v = \sqrt{-Ks}\epsilon\coth\left(\sqrt{\frac{-s}{2}}(x-ct)\right) - \sqrt{-Ks}\tanh\left(\sqrt{\frac{-s}{2}}(x-ct)\right) + \frac{c}{2\sqrt{2K}\epsilon} + \frac{c}{2\sqrt{2K}}, \end{cases}$$

(vi) $$\begin{cases} u = \sqrt{-Ks}\epsilon\tanh\left(\sqrt{\frac{-s}{2}}(x-ct)\right) + \sqrt{-Ks}\coth\left(\sqrt{\frac{-s}{2}}(x-ct)\right) - \frac{c}{4\sqrt{2K}\epsilon} + \frac{c}{4\sqrt{2K}}, \\ v = -\sqrt{-Ks}\epsilon\tanh\left(\sqrt{\frac{-s}{2}}(x-ct)\right) - \sqrt{-Ks}\coth(\sqrt{\frac{-s}{2}}(x-ct)) + \frac{c}{2\sqrt{2K}\epsilon} + \frac{c}{2\sqrt{2K}}. \end{cases}$$

In the case of $u_0 = 0$, $v_0 = 0$, solving Eq. (13) yields

$$u_{1,2} = \frac{c}{3\sqrt{2K}} \pm \sqrt{\frac{c^2}{9K} + \frac{B-Ks}{3}}, \quad v_{1,2} = -\frac{c}{3\sqrt{2K}} \pm 2\sqrt{\frac{c^2}{9K} + \frac{B-Ks}{3}},$$

$$u_2 = \begin{cases} \sqrt{\frac{K}{2}s}, & s > 0, \\ -\sqrt{\frac{K}{2}s}, & s < 0, \end{cases} \qquad v_2 = \begin{cases} -\frac{Ks}{\sqrt{2K}}, & s > 0, \\ \frac{Ks}{\sqrt{2K}}, & s < 0. \end{cases}$$

Similarly, we obtain the following soliton solutions of Eq. (10)

(vii) $$\begin{cases} u = \frac{c}{3\sqrt{2K}} \pm \sqrt{\frac{c^2}{9K} + \frac{B-Ks}{3}} + \sqrt{Ks}\tan\left(\sqrt{\frac{s}{2}}(x-ct)\right), \\ v = -\frac{c}{3\sqrt{2K}} \pm 2\sqrt{\frac{c^2}{9K} + \frac{B-Ks}{3}} - \sqrt{Ks}\tan\left(\sqrt{\frac{s}{2}}(x-ct)\right), s > 0, \end{cases}$$

(viii) $$\begin{cases} u = \frac{c}{3\sqrt{2K}} \pm \sqrt{\frac{c^2}{9K} + \frac{B-Ks}{3}} + \sqrt{Ks}\cot\left(\sqrt{\frac{s}{2}}(x-ct)\right), \\ v = -\frac{c}{3\sqrt{2K}} \pm 2\sqrt{\frac{c^2}{9K} + \frac{B-Ks}{3}} - \sqrt{Ks}\cot\left(\sqrt{\frac{s}{2}}(x-ct)\right), s > 0, \end{cases}$$

(ix) $$\begin{cases} u = \frac{c}{3\sqrt{2K}} \pm \sqrt{\frac{c^2}{9K} + \frac{B-Ks}{3}} - \sqrt{-Ks}\tanh\left(\sqrt{\frac{s}{2}}(x-ct)\right), \\ v = -\frac{c}{3\sqrt{2K}} \pm 2\sqrt{\frac{c^2}{9K} + \frac{B-Ks}{3}} + \sqrt{-Ks}\tanh\left(\sqrt{\frac{s}{2}}(x-ct)\right), s < 0, \end{cases}$$

(x) $$\begin{cases} u = \frac{c}{3\sqrt{2K}} \pm \sqrt{\frac{c^2}{9K} + \frac{B-Ks}{3}} - \sqrt{-Ks}\coth\left(\sqrt{\frac{s}{2}}(x-ct)\right), \\ v = -\frac{c}{3\sqrt{2K}} \pm 2\sqrt{\frac{c^2}{9K} + \frac{B-Ks}{3}} + \sqrt{-Ks}\coth\left(\sqrt{\frac{s}{2}}(x-ct)\right), s < 0. \end{cases}$$

Example 2. The Boussinesq equation

$$q_{tt} = q_{xx} + 3(q)^2_{xx} + q_{xxx} \tag{14}$$

was introduced by Boussinesq in 1871 to describe the propagation of long waves in shallow water (Weiss, 1985). This equation also arises in several other physical applications, including one-dimensional nonlinear string and ion sound wave in plasma. Therefore, it is of an important equation. Its some exact solutions and properties were found (Zhang, Y. F. and Zhang, H. Q., 2000). In this paper, some known and unknown solutions are given. To our best knowledge, the kinds of solutions like $\cot^2 \times \sec^2$, $\coth^2 \times\ \mathrm{sech}^2$, etc. are not discovered. To use conveniently the Conte expansion approach to get new soliton solutions for (14), a transformation is first applied to it.

Let $q = u_x$ and insert it into (14), we have

$$u_{tt} = u_{xx} + 6u_x u_{xx} + u_{xxxx}. \tag{15}$$

Set $u = \Sigma_{j=0}^{\infty} u_j v^{j+\alpha}$, then a leading term is expressed as $u_0 v^{\alpha}$. Similar to Eq. (10), we find that $\alpha = -1$. Hence, let

$$u = \frac{u_0}{v} + u_1 + u_2 v, \quad v_t = -c - \frac{1}{2}csv^2, \quad v_t = 1 + \frac{1}{2}sv^2. \tag{16}$$

Inserting (16) into (15) and setting all the coefficient powers of v to zero yield the following over-determined equations

$$\begin{cases} -12u_0^2 + 24u_0 = 0,\ -12u_0^2 s + 12u_0u_2 + 20u_0 s + 2u_0 = 2u_0c^2, \\ u_0c^2s = u_0 s - 3u_0^2 s^2 + 6u_0u_2 s + 4u_0 s^2,\ u_2c^2s = u_2 s - 3u_0u_2 s^2 + 6u_2^2 s + 4u_2 s^2, \\ \frac{1}{2}u_2c^2s^2 = \frac{1}{2}u_2 s^2 - \frac{3}{2}u_0u_2 s^3 + 6u_2^2 s^2 + 5u_2 s^3,\ u_2 s^4 + u_2^2 s^3 = 0. \end{cases} \tag{17}$$

Case 1. When $u_0 = 0$, we have from (17) that $u_2 = -s$, $c^2 = 1 - 2s$, u_1 is an arbitrary solution of (15). Thus, we obtain the periodic solutions and the bell-shape soliton solutions, respectively

$$\begin{cases} q = -s\ \sec^2\left(\sqrt{\frac{s}{2}}(x - ct)\right), \\ q = -s\,c\ \sec^2\left(\sqrt{\frac{s}{2}}(-x + ct)\right), s > 0, \end{cases} \begin{cases} q = s\,\mathrm{sech}^2\left(\sqrt{-\frac{s}{2}}(x - ct)\right), \\ q = -s\,c\,\mathrm{sech}^2\left(\sqrt{-\frac{s}{2}}(x - ct)\right), s < 0, \end{cases}$$

Case 2. When $u_0 = 2$, in terms of (17), we have $u_2 = -s$, $c^2 = 1 - 8s$, u_1 is arbitrary. Thus, we give the following soliton solutions of Eq. (14)

$$\begin{cases} q = -s\ \cot^2\left(\sqrt{\frac{s}{2}}(x - ct)\right) \sec^2\left(\sqrt{\frac{s}{2}}(x - ct)\right) - s\ \sec^2\left(\sqrt{\frac{s}{2}}(x - ct)\right), \quad s > 0, \\ q = -s\ \tan^2\left(\sqrt{\frac{s}{2}}(-x + ct)\right) c\ \sec^2\left(\sqrt{\frac{s}{2}}(-x + ct)\right) + s\,c\ \sec^2\left(\sqrt{\frac{s}{2}}(-x + ct)\right), \quad s > 0, \end{cases}$$

$$\begin{cases} q = -s\coth^2\left(\sqrt{\frac{-s}{2}}(x - ct)\right) c\,\mathrm{sech}^2\left(\sqrt{\frac{-s}{2}}(x - ct)\right) + s\,\mathrm{sech}^2\left(\sqrt{\frac{-s}{2}}(x - ct)\right), \quad s < 0, \\ q = -s\tanh^2\left(\sqrt{\frac{-s}{2}}(x - ct)\right) \mathrm{sech}^2\left(\sqrt{\frac{-s}{2}}(x - ct)\right) + s\,c\,\mathrm{sech}^2\left(\sqrt{\frac{-s}{2}}(x - ct)\right), \quad s < 0. \end{cases}$$

Example 3. Consider the modified Burgers equation (Lee-Bapty and Crighton, 1987)

$$u_t + qu^2u_x + pu_{xx} = 0, \tag{18}$$

where p, q are constants. If we let $u = \Sigma_{j=0}^{\infty} u_j v^{j+\alpha}$, then using leading term analysis, we $\alpha = -\frac{1}{2}$. Thus, we can not use directly the Conte expansion to solve Eq. (18). But let $u = \sqrt{w}$ and substitute it into (18), we have

$$ww_t + qw^2w_x + p\left(ww_{xx} - \frac{1}{2}w_x^2\right) = 0. \tag{19}$$

Set $w = \Sigma_{j=0}^{\infty} w_j v^{j+\alpha}$, we can find that $\alpha = -1$ (integer). Thus, let

$$w = \frac{w_0}{v} + w_1 + w_2 v, \quad v_t = -c - \frac{1}{2}csv^2, \quad v_x = 1 + \frac{1}{2}sv^2. \tag{20}$$

Substituting (20) into (19) and vanishing the coefficients of the different powers of v leads to

$$\begin{cases} w_0\left(2pw_0 - qw_0^2\right) - \frac{p}{2}w_0^2 = 0, w_0^2\,(c - w_1 q) + w_1\left(2pw_0 - qw_0^2\right) = 0, \\ w_0\left(w_0 ps - \frac{1}{2}w_0^2 qs\right) + w_1\,(w_0 c - w_0 w_1 q) + w_2\left(2pw_0 - qw_0^2\right) + pw_0w_2 - \frac{1}{2}psw_0^2 = 0, \\ w_0\left(\frac{1}{2}w_0 cs - w_2 c + qw_1w_2 - \frac{1}{2}w_0w_1qs\right) + w_1\left(w_0 ps - \frac{1}{2}w_0^2 qs\right) + w_2(w_0c - w_0w_1q) = 0) \\ w_0\left(w_2^2 q + w_2 ps\right) + w_1\left(\frac{1}{2}w_0cs - w_2c + qw_1w_2 - \frac{1}{2}qsw_0w_1\right) \\ \quad + w_2\left(w_0 ps - \frac{1}{2}w_0^2 qs\right) + \frac{1}{2}psw_0w_2 - \frac{p}{2}\left(w_2 - \frac{1}{2}w_0 s\right)^2 = 0, \\ w_0\left(\frac{1}{2}qsw_1w_2 - \frac{1}{2}w_2cs\right) + w_1\left(w_2^2 q + w_2 ps\right) + w_2\left(\frac{1}{2}w_0cs - w_2c + qw_1w_2 - \frac{1}{2}qsw_0w_1\right) = 0, \\ \frac{1}{2}w_0\left(w_2^2 qs + w_2 ps^2\right) + \frac{1}{2}w_1\,(qsw_1w_2 - csw_2) + w_2\left(w_2^2 q + w_2 ps\right) - \frac{1}{2}psw_2^2 + \frac{1}{4}ps^2w_0w_2 = 0, \\ w_2\left(qsw_1w_2 - w_2ps^2\right) + w_1\left(w_2^2 qs + w_2 ps^2\right) = 0, w_2\left(w_2^2 qs + w_2 ps^2\right) - \frac{p}{4}w_2^2 s^2 = 0. \end{cases} \tag{21}$$

When $w_0 \neq 0$, we obtain a set of solutions of (21): $w_0 = \frac{3p}{2q}$, $w_1 = \frac{3c}{2q}$, $w_2 = \frac{-3ps}{4q}$, where $8c^2 + p^2 s = 0$. Hence, when $s > 0$, we have the periodic solutions of (19)

$$w = \frac{3p}{2q}\sqrt{\frac{s}{2}}\cot\left(\sqrt{\frac{s}{2}}(x - ct)\right) - \frac{3ps}{4q}\sqrt{\frac{2}{s}}\tan\left(\sqrt{\frac{s}{2}}(x - ct)\right) + \frac{3c}{2q}, \tag{22}$$

$$w = \frac{3p}{2q}\sqrt{\frac{s}{2}}\tan\left(\sqrt{\frac{s}{2}}(-x + ct)\right) - \frac{3ps}{4q}\sqrt{\frac{2}{s}}\cot\left(\sqrt{\frac{s}{2}}(-x + ct)\right) + \frac{3c}{2q}. \tag{23}$$

When $s < 0$, the singular solutions are obtained as follows

$$w = \frac{3p}{2q}\sqrt{\frac{-s}{2}}\coth\left(\sqrt{\frac{-s}{2}}(x - ct)\right) - \frac{3ps}{4q}\sqrt{\frac{-2}{s}}\tanh\left(\sqrt{\frac{-s}{2}}(x - ct)\right) + \frac{3c}{2q}, \tag{24}$$

$$w = \frac{3p}{2q}\sqrt{\frac{-s}{2}}\tanh\left(\sqrt{\frac{-s}{2}}(x-ct)\right) - \frac{3ps}{4q}\sqrt{\frac{-2}{s}}\coth\left(\sqrt{\frac{-s}{2}}(x-ct)\right) + \frac{3c}{2q}. \tag{25}$$

In the case of $w_0 = 0$, we have a set of solutions of (21) $w_1 = \frac{3c}{2q}$, $w_2 = -\frac{3ps}{4q}$, $c^2 + p^2 s = 0$. Similar to (22)–(25), we also have the exact solutions of (19). Here we omit them. Thus, using the transformation $u = \sqrt{w}$ can obtain the corresponding exact solutions of Eq. (18).

Remark. Zhang (2001) only obtained the part solution of (24), i.e., the form solution like $w = -\frac{3ps}{4q}\sqrt{-\frac{2}{s}}\tanh(\sqrt{-\frac{s}{2}}(x-ct)) + \frac{3c}{2q}$ was presented. Obviously, here we extend largely the results in Zhang (2001).

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